Polylogarithms Of Order Zero

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In this white paper we will examine the polylogarithm in the following form...

$$Li_{-s}(z)=\sum_{k=1}^{\infty}k^{s}z^{k}\ \ \text{...where...}\ \ s\in\{0,1,2,3,4,...\}\ \ \text{...and...}\ \ |z|<1 \tag{1}$$

When the parameter s (order) in Equation (1) above is equal to zero then the equation for a polylogarithm of order zero is...

$$Li_{0}(z)=\sum_{k=1}^{\infty}z^{k}\ \ \text{...where...}\ \ |z|<1 \tag{2}$$

Our Hypothetical Problem

Given that the parameter $z=0.80$ and the parameter $s=0$ then answer the following questions...

1. What is the value of the polylogarithm over the interval $k=1$ to infinity?

2. What is the value of the polylogarithm over the interval $k=1$ to 4?

Building the Equations

Using Equation (2) above and Appendix Equation (10) below the equation for the value of a polylogarithm of order zero over the interval $k=1$ to $k=\infty$ is..

$$Li_{0}(z)=\sum_{k=1}^{\infty}z^{k}=z\frac{\delta Li_{-1}(z)}{\delta z}=z\,\frac{1}{1-z}=\frac{z}{1-z} \tag{3}$$

Using Equation (3) above the equation for the value of a polylogarithm of order zero over the interval $k=1$ to n is...

$$\sum_{k=1}^{n}z^{k}=\sum_{k=1}^{\infty}z^{k}-\sum_{k=n+1}^{\infty}z^{k} \tag{4}$$

Note that we can rewrite the third term in Equation (4) above as...

$$\sum_{k=n+1}^{\infty}z^{k}=z^{n}\sum_{k=1}^{\infty}z^{k} \tag{5}$$

Using Equation (3) above we can rewrite Equation (5) above as...

$$z^{n}\sum_{k=1}^{\infty}z^{k}=z^{n}\frac{z}{1-z}=\frac{z^{n+1}}{1-z} \tag{6}$$

Using Equations (3) and (6) above we can rewrite Equation (4) above as...

$$\sum_{k=1}^{n}z^{k}=\frac{z}{1-z}-\frac{z^{n+1}}{1-z}=\frac{z-z^{n+1}}{1-z} \tag{7}$$

The Answers To Our Hypothetical Problem

1. What is the value of the polylogarithm over the interval $k = 1$ to infinity?

Using Equation (3) above the answer to the question is...

$$\sum_{k=1}^{\infty} 0.80^k = \frac{0.80}{1 - 0.80} = 4.00 \tag{8}$$

2. What is the value of the polylogarithm over the interval $k = 1$ to 4?

Using Equation (7) above the answer to the question is...

$$\sum_{k=1}^{4} 0.80^k = \frac{0.80 - 0.80^5}{1 - 0.80} = 2.36 \tag{9}$$

Appendix

A. The equation for the base polylogarithm is...

$$Li_1 z = \sum_{k=1}^{\infty} k^{-1} z^k = -ln(1 - z) \ \text{...where...} \ \frac{\delta Li_{-1}(z)}{\delta z} = \frac{1}{1 - z} \tag{10}$$

B. Using Equation (3) above the solution to the following summation is...

$$\begin{aligned} \sum_{k=0}^{\infty} z^k &= \left[\sum_{k=1}^{\infty} z^k \right] + 1 \\ &= \frac{z}{1 - z} + \frac{1 - z}{1 - z} \\ &= \frac{z + 1 - z}{1 - z} \\ &= \frac{1}{1 - z} \ \text{...where...} \ |z| < 1 \end{aligned} \tag{11}$$